Algebra of Matrices

Pravin K. Verekar, Ph.D.(IISc)

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`Sir Isaac Newton: If I have seen further, it is by standing on the shoulders of giants.`

1 Matrices

A matrix is a rectangular arrangement of elements (entries) in rows and columns. A matrix is usually denoted by capital letter and the entries in the matrix are denoted by lower case letters. Each element belongs to a row and a column of the matrix. A matrix A has element a_{ij} in the i-th row and j-th column. A matrix A with m horizontal rows and n vertical columns has the following form.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Pravin K. Verekar
La Campala Colony
Miramar, Panaji, Goa
India, Pin Code 403001
E-mail: prof.verekar@gmail.com

This matrix A is a m *by* n matrix, written $m \times n$. The pair of numbers m and n define the *size* of the matrix. We frequently denote such a matrix by simply writing $A = [a_{ij}]$.

A matrix with only one row is called a *row matrix* or *row vector*, and a matrix with only one column is called a *column matrix* or *column vector*. We can look at the $m \times n$ matrix as consisting of n column vectors or as consisting of m row vectors. The matrix A then can be given by appropriate column vectors a_1, a_2, $\cdots$, a_n,

$$A = \begin{bmatrix} a_1 \; a_1 \cdots a_n \end{bmatrix}$$

or by appropriate row vectors a_1, a_2, $\cdots$, a_m.

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

A matrix whose entries are all zero is called a *zero matrix* and will usually be denoted by 0.

Matrices whose entries are all real numbers are called *real matrices* and are said to be *matrices over* $\mathbf{R}$. Matrices whose entries are all complex numbers are called *complex matrices* and are said to be *matrices over* $\mathbf{C}$.

Two matrices A and B are *equal*, written $A = B$, if they have the same size and if corresponding elements are equal.

2 Matrix Addition and Scalar Multiplication

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ with the same size, say $m \times n$ matrices, are added as shown below.

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

The multiplication of a matrix A by a scalar k is obtained by multiplying each element of A by k as shown below.

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}$$

3 Matrix Multiplication

The multiplication of two matrices A and B, written AB, is possible only if the number of columns of matrix A is equal to the number of rows of matrix B; so, say, A is an $m \times p$ matrix and B is an $p \times n$ matrix. Then the product AB, say equal to matrix C, is the $m \times n$ matrix whose ij-th entry is obtained by multiplying the ith row of A by the jth column of B as shown below.

$$\begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \cdot & \cdots & \cdot \\ a_{i1} & \cdots & a_{ip} \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & c_{ij} & \cdot \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$

Thus, matrix multiplication is a multiplication of rows of the prefactor matrix A into columns of the postfactor matrix B.

So, matrix multiplication depends on the *shapes* of the matrices. For two appropriate rectangular matrices A and B that give product AB, it is most likely that product BA may not be defined. Even for square matrices A and B for whom it is possible to find products AB and BA, usually $AB \neq BA$. The general rule for matrix multiplication is that *matrix multiplication is not commutative.*

Let A, B, C be matrices. Then, whenever the products and sums are defined, the following hold true.

1. $(AB)C = A(BC)$ (associative law),
2. $A(B + C) = AB + AC)$ (left distributive law),
3. $(B + C)A = BA + CA)$ (right distributive law),
4. $k(AB) = (kA)B + A(kB)$, where k is a scalar.

Note that $0A = 0$ and $B0 = 0$, where 0 is the zero matrix.

4 Transpose of a Matrix

The *transpose* of a matrix A, written A^T, is obtained by writing the columns of A as rows; this is the same as writing the rows of A as columns. The jth column of A becomes the jth row of A^T. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -3 & -5 \end{bmatrix}^T = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}$$

If A is an $m \times n$ matrix, then A^T is the $n \times m$ matrix. Let B be equal to A^T, then the entries $b_{ij} = a_{ji}$.

Observe that the transpose of a row vector is a column vector. Similarly, the transpose of a column vector is a row vector.

The basic properties of the transpose operation are listed below. Let A and B be matrices and let k be a scalar. Then, whenever the sum and product are defined:

1. $(A + B)^T = A^T + B^T$,
2. $(A^T)^T = A$,
3. $(kA)^T = kA^T$,
4. $(AB)^T = B^T A^T$.

From the last relation, note that the transpose of a product is the product of the transposes of the individual matrices in the reverse order.

5 Square Matrices

A square matrix has same number of rows and columns. An $n \times n$ square matrix is said to be of *order* n and is sometimes called an *n-square matrix*. The beauty of square matrices is that any matrices of the same order can be added or multiplied. A word of caution however, matrix multiplication need not be commutative.

5.1 Diagonal and Trace

For an $n \times n$ square matrix $A = [a_{ij}]$, the elements a_{11}, a_{22}, a_{33}, $\ldots$, a_{nn} form the *diagonal* or *main diagonal* of A. The trace of A is the sum of the diagonal entries.

$$\text{trace } A = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$$

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be n-square matrices and k be a scalar. The following rules apply.

1. $\text{trace}\,(A + B) = \text{trace}\,(A) + \text{trace}\,(B)$,
2. $\text{trace}\,(kA) = k\ \text{trace}\,(A)$,
3. $\text{trace}\,(A^T) = \text{trace}\,(A)$,
4. $\text{trace}\,(AB) = \text{trace}\,(BA)$.

5.2 Identity Matrix, Scalar Matrices

The *identity* or *unit* matrix is an n-square matrix, written I_n or simply I, with 1's on the diagonal and 0's elsewhere. For any n-square matrix A, we have

$$AI = IA = A.$$

More generally, if B is an $m \times n$ matrix, then

$$BI_n = I_m B = B.$$

The multiplication of any matrix with the identity matrix leaves it unchanged.

The product of any scalar k and the identity matrix I gives the matrix kI; it contains k's on the diagonal and 0's elsewhere; this matrix is called the *scalar matrix*. Note that

$$(kI)A = k(IA) = kA$$

That is, multiplying a matrix A by the scalar matrix kI is equivalent to multiplying A by the scalar k.

It is a common practice to omit blocks or patterns of 0's when there is no ambiguity; see examples below.

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 7 & & & \\ & 7 & & \\ & & 7 & \\ & & & 7 \end{bmatrix}$$

The *Kronecker delta function* δ_{ij} is defined as

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Then the unit matrix can be written as $I = [\delta_{ij}]$.

5.3 Diagonal and Triangular Matrices

A square matrix $D = [d_{ij}]$ is *diagonal* if its nondiagonal entries are all zero. Such a matrix is sometimes denoted by

$$D = \text{diag}(d_{11}, d_{22}, \ldots, d_{nn})$$

where some or all or none of the d_{ii} may be zero.

A square matrix $A = [a_{ij}]$ is *upper triangular* or simply *triangular* if all the entries below the diagonal are equal to 0, that is, if $a_{ij} = 0$ for $i > j$.

A square matrix whose elements above the diagonal are all zero is called *lower triangular matrix*.

6 Invertible (Nonsingular) Matrices

A square $n \times n$ matrix A is said to be *invertible* or *nonsingular* if there exists an $n \times n$ matrix B such that

$$AB = BA = I$$

where I is the identity matrix. It can be shown that such a matrix B is unique. If $AB_1 = B_1A = I$ and $AB_2 = B_2A = I$, then

$$B_1 = B_1I = B_1(AB_2) = (B_1A)B_2 = IB_2 = B_2$$

We call such a matrix B the *inverse* of A and denote it by A^{-1}. Note that if B is inverse of A, then A is inverse of B.

For $n \times n$ invertible matrices A and B, we have $(AB)^{-1} = B^{-1}A^{-1}$. A similar rule holds with three or more matrices: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

It can be shown that $(A^{-1})^T = (A^T)^{-1}$ for an invertible matrix A. So, $(A^{-1})^T A^T = I$.

A square matrix whose inverse does not exist is called a *singular matrix*. The determinant of a singular matrix is always zero. The determinant of a nonsingular matrix never vanishes, i.e. never equal to zero.

7 Special Real Square Matrices: Symmetric, Orthogonal, Normal

7.1 Symmetric and Skew-symmetric Matrices

Suppose A is a square matrix with real entries, that is, a real square matrix. The matrix A is a *symmetric* matrix if $A^T = A$. Thus, a symmetric is a matrix that equals its own transpose. Each entry on one side of the diagonal equals its "mirror image" on the on the other side, that is, $a_{ij} = a_{ji}$.

A symmetric matrix need not be invertible; it could even be a matrix of zeros. *But if A^{-1} exits, it is also symmetrical.*

Let R be any real rectangular $m \times n$ matrix. Then the product R^TR is a square $n \times n$ symmetric matrix. The product RR^T is also symmetric; it is a square $m \times m$ matrix which is different from R^TR. Even if $m = n$, that is, if R is a square matrix, it is very likely that $R^TR \neq RR^T$.

Strang (2006): Most scientific problems that start with a rectangular matrix R end up with R^TR or RR^T or both. Symmetric matrices appear in every subject

whose laws are fair. "Each action has an equal and opposite reaction." The entry a_{ij} that gives the action of i onto j is matched by a_{ji}.

A matrix A is *skew-symmetric* or *antisymmetric* if $A^T = -A$, that is, if each $a_{ij} = -a_{ji}$. The diagonal entries of such a matrix must be zero, since $a_{ii} = -a_{ii}$ implies $a_{ii} = 0$.

7.2 Orthogonal Matrices

A real matrix A is *orthogonal* if $A^T = A^{-1}$, that is, if $AA^T = A^TA = I$. Thus A must necessarily be square and invertible.

The vectors $u_1, u_2, \ldots, u_m$ in $\mathbf{R}^n$ are said to form an *orthonormal set* of vectors if the vectors are unit vectors and are orthogonal to each other, that is, $u_i \cdot u_j = \delta_{ij}$ where δ_{ij} is the Kronecker delta function.

For an orthogonal matrix A, the condition $AA^T = I$ implies that the rows of A form an orthonormal set of vectors, and the condition $A^TA = I$ similarly implies that the columns of A form an orthonormal set of vectors.

IN BRIEF:
A square matrix A is called

symmetric	$A^T = A$
skew-symmetric	$A^T = -A$
orthogonal	$A^T = A^{-1}$

7.3 Normal Matrices

A real matrix A is *normal* if it *commutes* with its transpose A^T, that is, if $AA^T = A^TA$. If A is symmetric, orthogonal, or skew-symmetric, then A is normal. There are also other normal matrices.

8 Complex Matrices

A matrix with complex entries is called a complex matrix. The three classes of matrices, namely, symmetrical, skew-symmetric and orthogonal matrices, discussed in Section 7, have complex counterparts, namely, *Hermitian*, *skew-Hermitian* and *unitary* matrices, which are of practical importance in certain applications, for instance, in quantum mechanics.

Let A be any complex rectangular matrix, that is, it has complex entries a_{ij}. Recall that if $z = a + bi$ is a complex number, then $\bar{z} = a - bi$ is its conjugate. The *conjugate* of the complex matrix A, written $\bar{A}$, is the matrix obtained from A by taking the conjugate of each entry in A. So the entries in $\bar{A}$ are $\bar{a}_{ij}$. Some text books denote the operation of conjugation by superscript $*$, in that case complex conjugate is denoted by A^* which same as $\bar{A}$.

Taking the transpose of the *complex conjugate* $\bar{A}$, we get the conjugate transpose of A, that is, $\bar{A}^T$. The two operations of transpose and conjugation commute for any complex matrix. That is,

$$\bar{A}^T = \bar{A^T}$$

Some text books denote the combined operation of transpose and conjugation by superscript H or $\dagger$, and call the operation *Hermitian*. (This is not to be confused with Hermitian matrix.) So, conjugate transpose or Hermitian is $\bar{A}^T = A^H = A^\dagger$.

For the complex rectangular matrices $A_{m\times p}$ and $B_{p\times n}$, the product $(AB)^H = B^HA^H$.

For a complex square matrix A, we define

Hermitian matrix	if $\bar{A}^T = A$
skew-Hermitian matrix	if $\bar{A}^T = -A$
unitary matrix	if $\bar{A}^T = A^{-1}$

Thus, $A = [a_{ij}]$ is Hermitian if and only if symmetric elements are conjugate,that is, if each $a_{ij} = \bar{a}_{ij}$, in which case each diagonal element a_{ii} must be real. Similarly, if $A = [a_{ij}]$ is skew-Hermitian, each diagonal element $a_{ii} = 0$.

Inner Product (Dot Product):
The inner product between two real vectors x and y is given by x^Ty. The length of the vector x is found from the equation $\|x\|^2 = x^Tx$.

The inner product between two complex vectors x and y is given by $\bar{x}^Ty$, also written as, x^Hy. If the two vectors are orthogonal then $x^Hy = 0$. The squared length of vector x is $\|x\|^2 = x^Hx$. For the column vector x, written horizontally as, $(x_1, \ldots, x_n)$, x^H is the row vector $[\bar{x}_1 \ \ldots \ \bar{x}_n]$.

A complex square invertible matrix A is *unitary* if $A^HA^{-1} = A^{-1}A^H = I$, that is, if $A^H = A^{-1}$. We note that a complex matrix A is unitary if and only if its row vectors (and also column vectors) form an orthonormal set relative to the dot product of complex vectors.

In the absence of imaginary part in the entries of Hermitian, skew-Hermitian and unitary matrices, these matrices become symmetric, skew-symmetric and orthogonal matrices respectively.

A complex square matrix A is *normal* if it commutes with A^H, that is, if $AA^H = A^HA$.

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